Mortality above age 105
New data, new models

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EAPS Working Group "Health, Morbidity and Mortality" workshop

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[www.supercentenarians.org]

  - individual data on deaths 105+, i.e. *semi-supercentenarians*
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- !! Recent challenge in obtaining new individual data !!

Aim of the study

Research questions

1. map out time trends for semi-supercentenarians
2. unravel the conundrum of the mortality plateau
3. describe the age pattern of mortality without imposing any hypotheses
4. assess eventual sex and cohort differences in mortality above age 105

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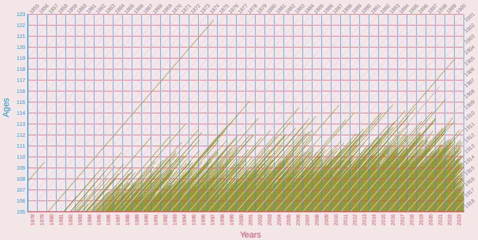
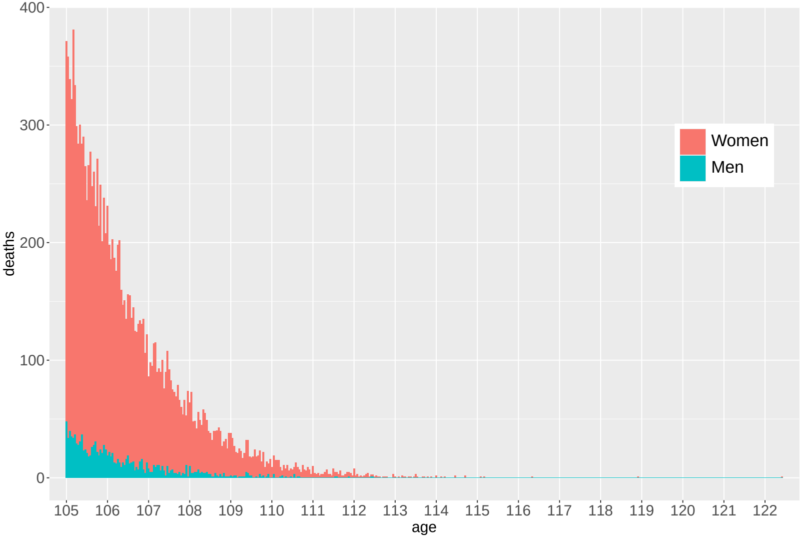
Data

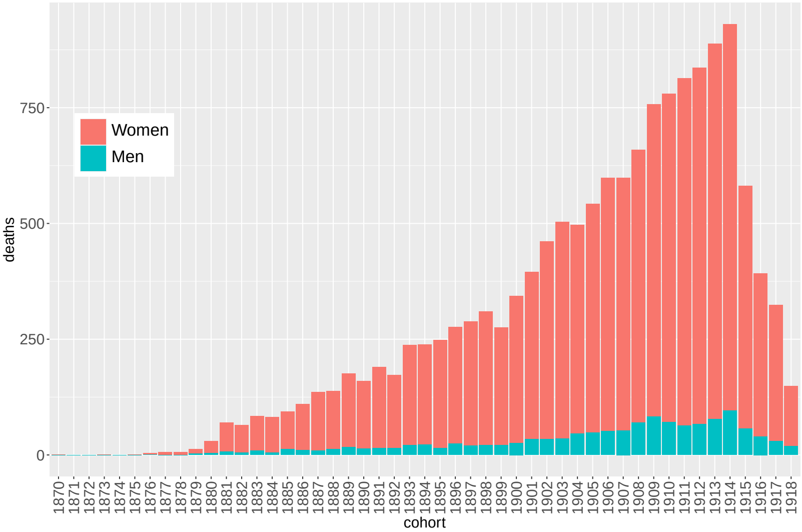
- We have recently received updated data from France



- 14,467 Deaths 105+
- Women 91% – Men 9%
- Years: 1978-2023
- Cohorts: 1870-1918

Some descriptive facts: deaths by age




Women
Men
deaths
750
500
250
0
cohort
1870
1871
1872
1873
1874
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1878
1879
1880
1881
1882
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1911
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1914
1915
1916
1917
1918

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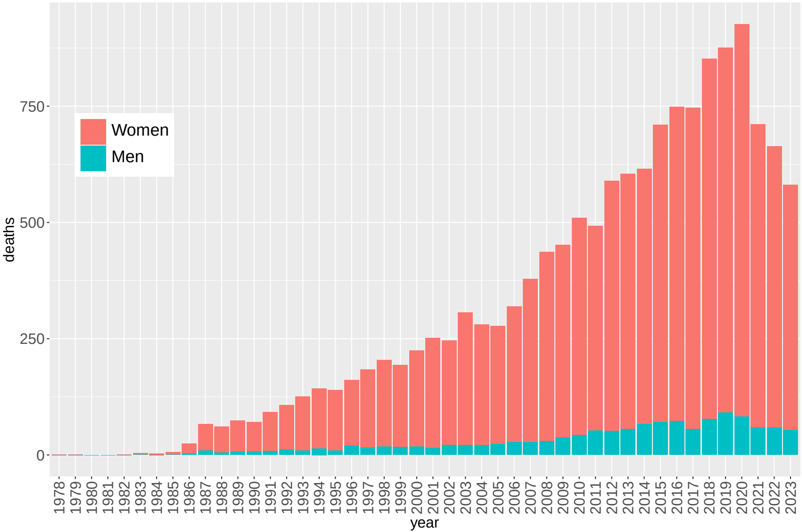

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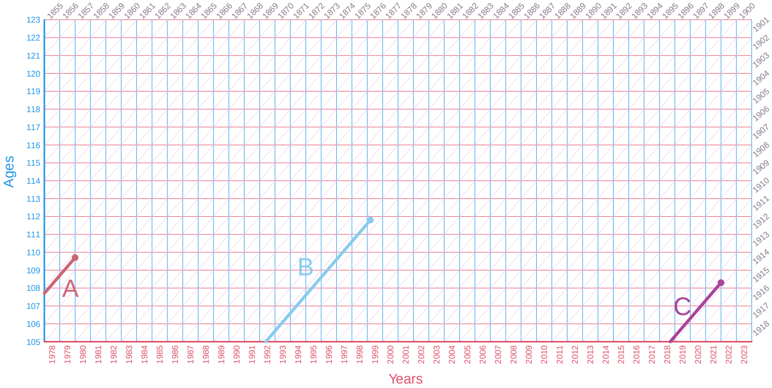
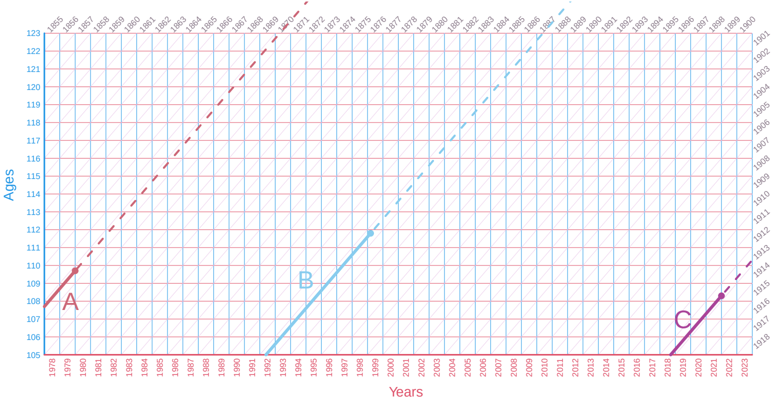
  - **Right Truncation**: Only individuals who died by the end of 2023 are included
    - Potential bias from incomplete population representation: Exclusion of individuals who may die after 2023, as they will only be included once they pass away
    - In practice, for earlier cohorts, all deaths are observed, i.e., extinct cohorts



	Date of birth	Date of death	Age at death	Date of entry	Entry time
A	1870.3	1980.0	109.7	1978.0	107.7
B	1887.4	1999.2	111.8	1992.4	105.0
C	1913.7	2022.0	108.3	2018.7	105.0

Model the complete dataset



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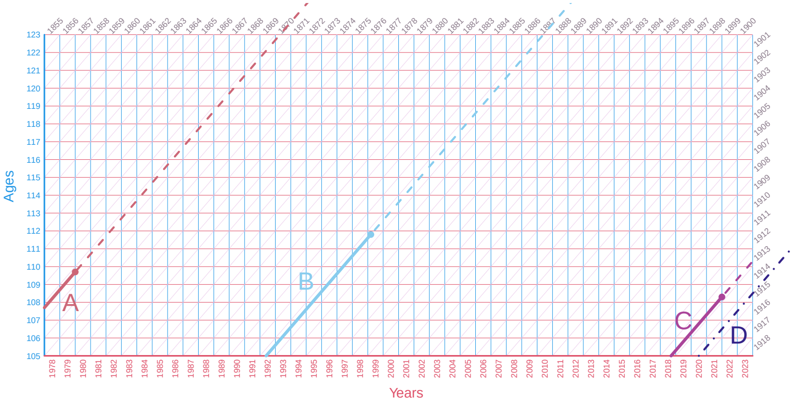

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D	1915.5	?	?	2020.5	105.0	–

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$$\ell_A = P(X = 4.7 \mid X > 2.7) = \frac{f(4.7)}{S(2.7)}$$

$$\ell_B = P(X = 6.8) = f(6.8)$$

$$\ell_C = P(X = 3.3 \mid X \leq 5.3) = \frac{f(3.3)}{1 - S(5.3)}$$

- We can safely assume that $S(153.7)$ and $S(136.6)$ are both equal to 0
- We can condition everything on surviving to age 105

Generalizing the assumption

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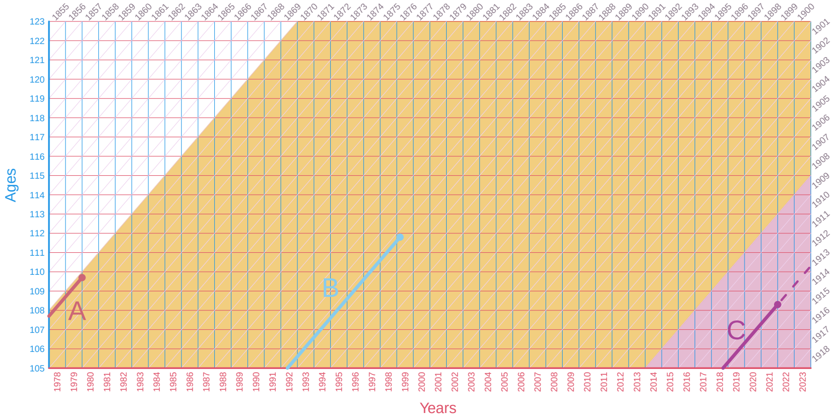
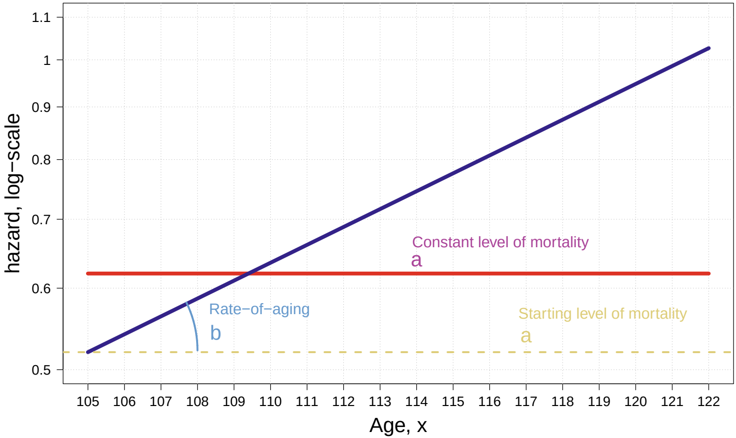


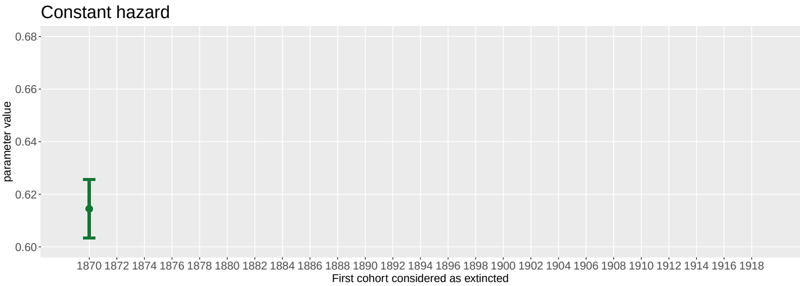
Cohorts: 1870–1908 Right truncation can be safely disregarded

Cohorts: 1909–1918 Right truncation needs to be accounted for

- Two main options for the mortality hazard:
  - Constant : $h(x) = a$
  - Gompertz: $h(x) = a\, e^{bx}$



Effect of disregarding right truncation



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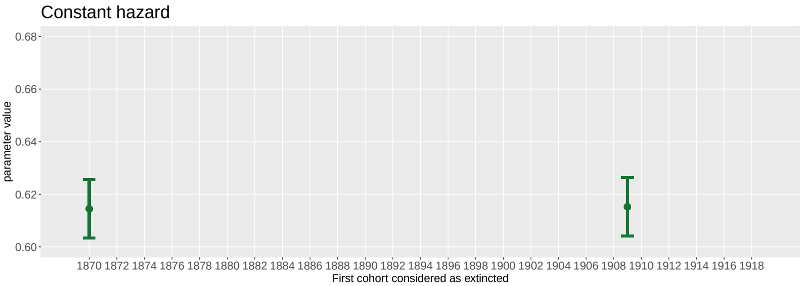

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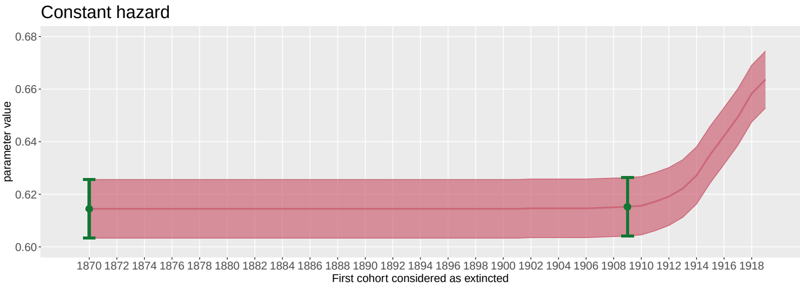

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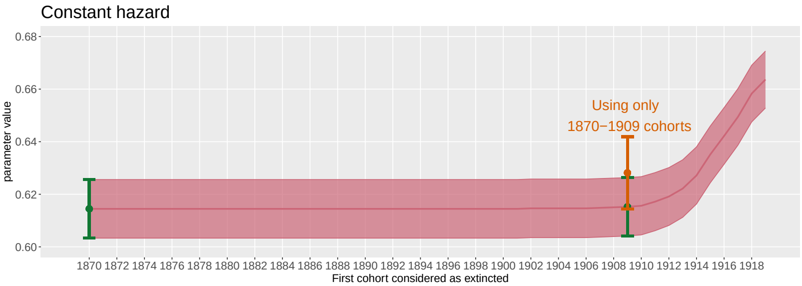

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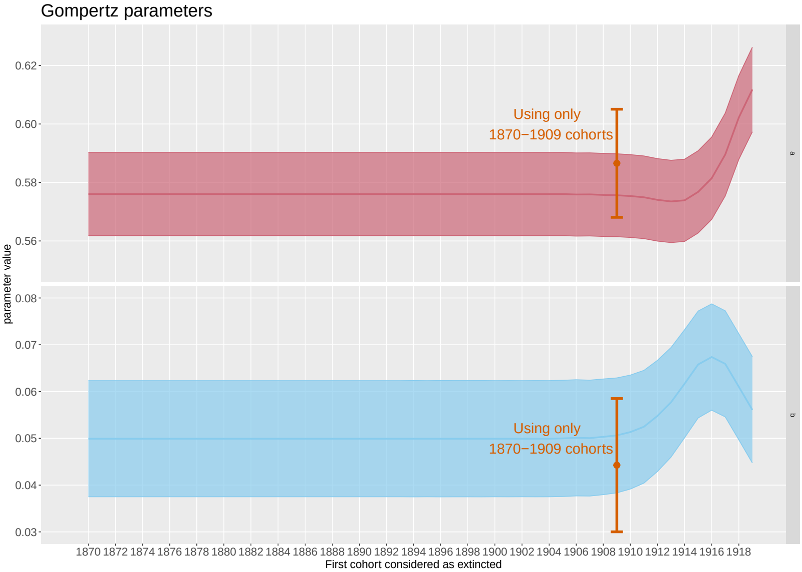

Including sex and/or cohort

- Check for sex differences in a proportional hazards setting:

$$h_i(x \mid \text{sex}_i, \text{cohort}_i) = h_0(x) \cdot \begin{cases} e^{\beta_{sex} \cdot \text{sex}_i} \\ e^{\beta_{cohort} \cdot \text{cohort}_i} \\ e^{\beta_{sex} \cdot \text{sex}_i + \beta_{cohort} \cdot \text{cohort}_i} \end{cases}$$

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- Gompertz baseline with only sex as a covariate selected by AIC

Parameter	Estimated	95% CI
a	0.56572	[0.55132, 0.58011]
b	0.05231	[0.04001, 0.06461]
β_{sex}	0.18511	[0.12291, 0.24730]

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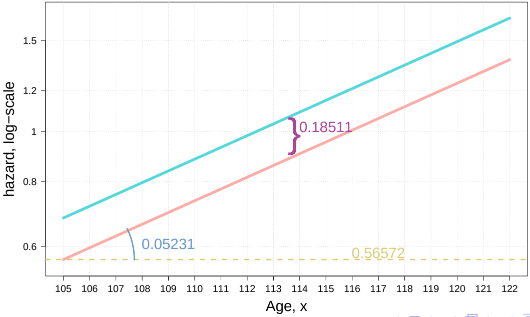

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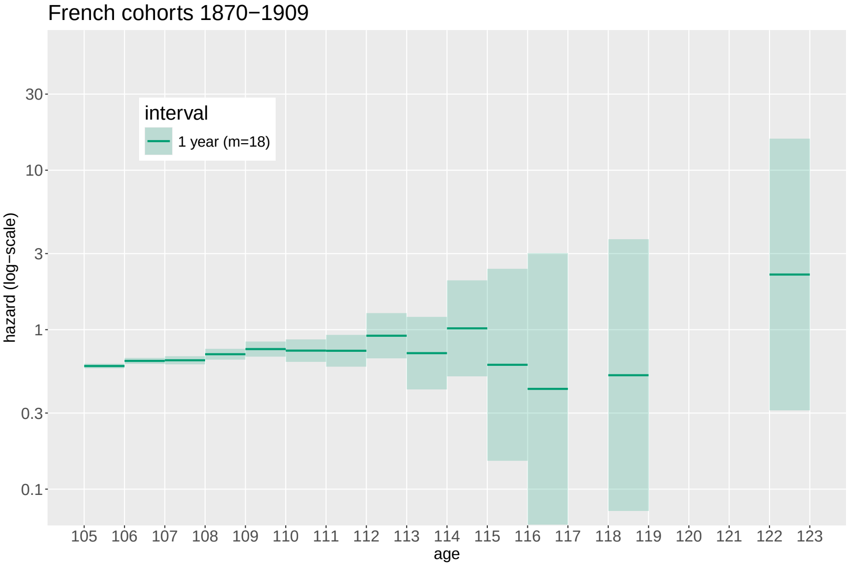
- Issues:
  - estimated hazard function will be discontinuous
  - subjectivity in the choice of the breakpoints

Piecewise constant hazard model in action



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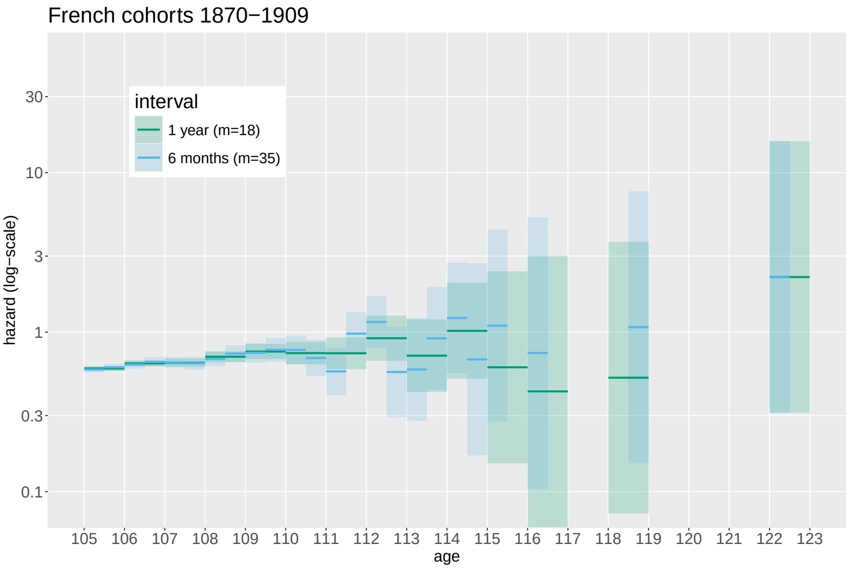

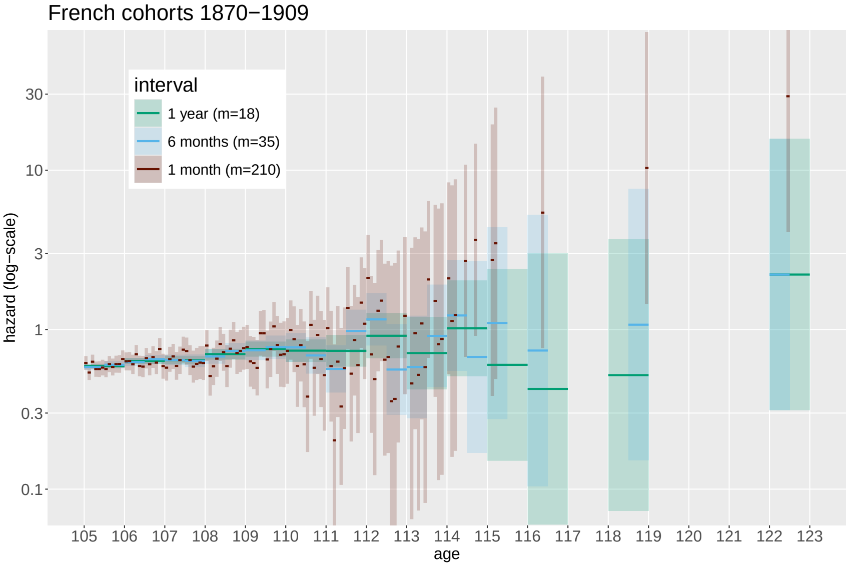
French cohorts 1870−1909
interval
1 year (m=18)
6 months (m=35)
1 month (m=210)
hazard (log−scale)
30
10
3
1
0.3
0.1
105
106
107
108
109
110
111
112
113
114
115
116
117
118
119
120
121
122
123
age

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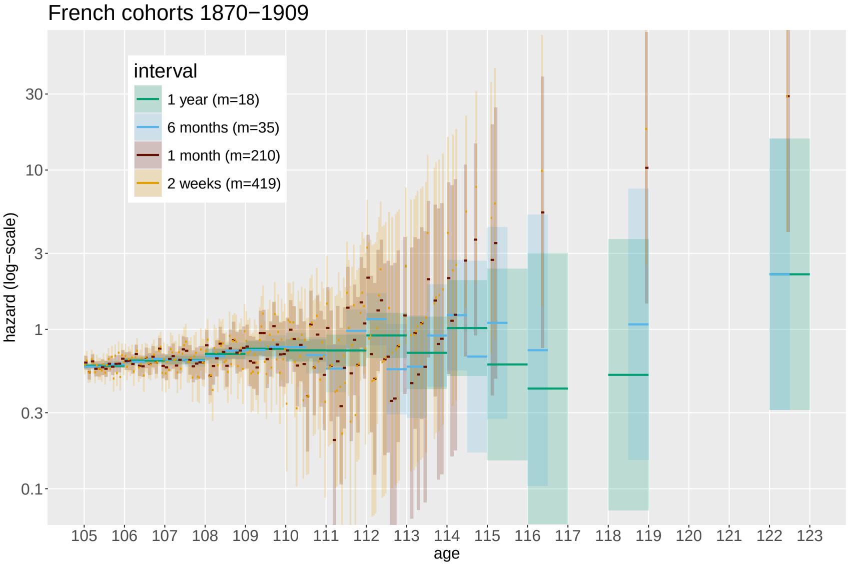

Can we do better? Yes!

- Recipe of estimate a non-parametric hazard:
  - Divide the age axis into *many* equally spaced intervals
  - Construct y_j (observed events) and e_j (exposure) in each interval
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- An equation for an iterative algorithm

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  - Number of intervals is irrelevant if m is large
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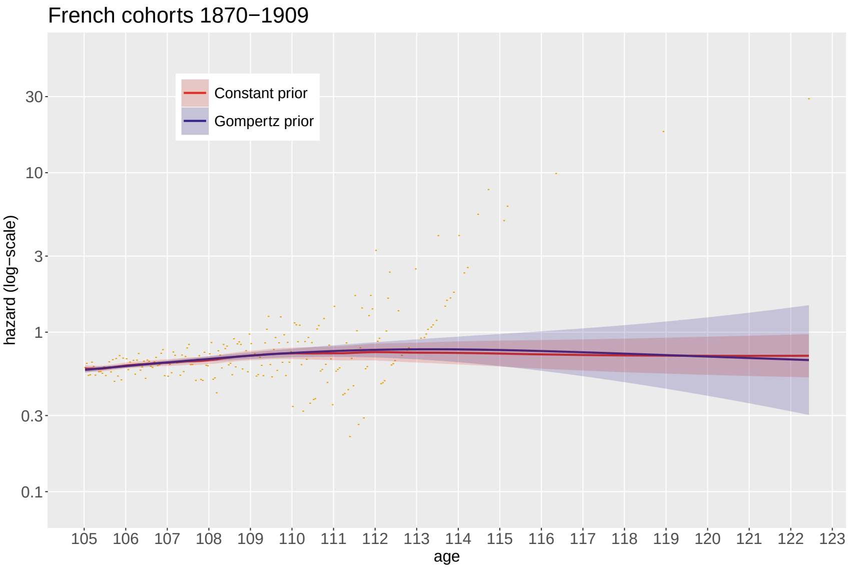
- Current issue: we cannot deal with right truncation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Gompertz prior
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3
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age

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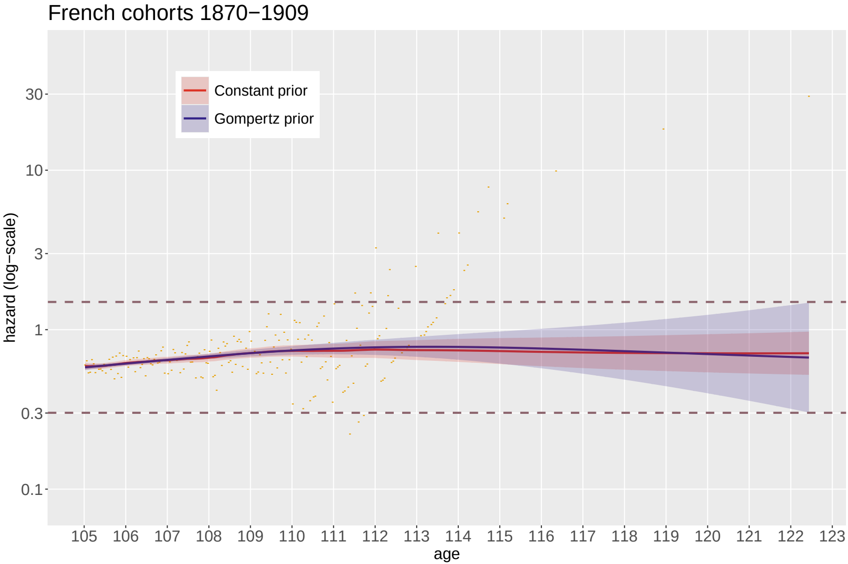
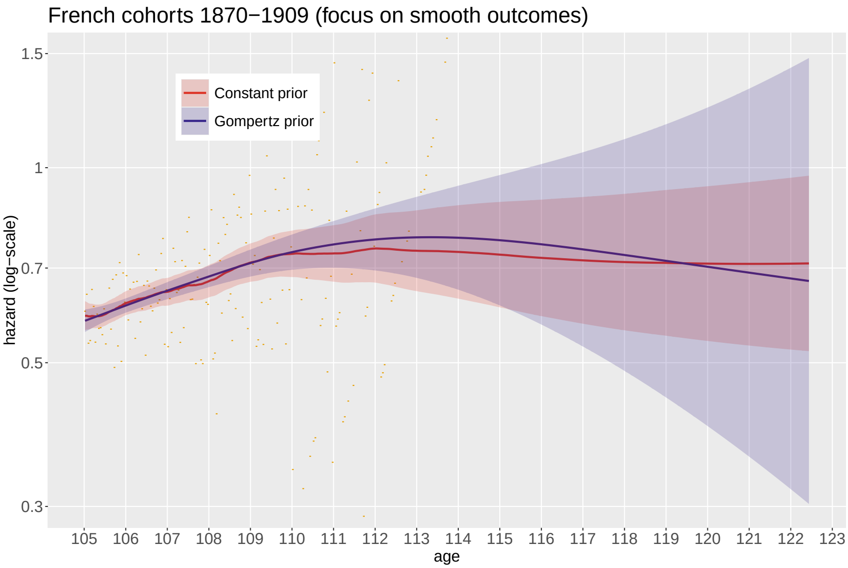


Non-parametric hazard in action



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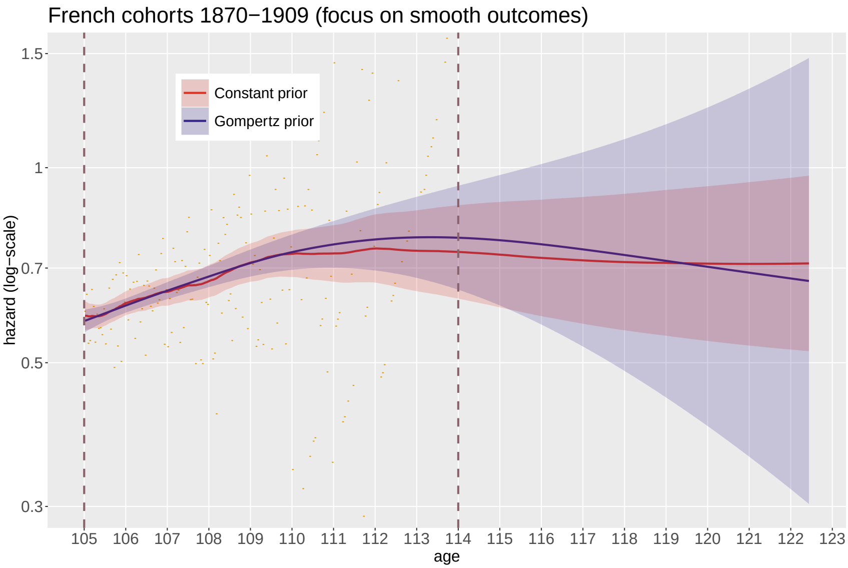

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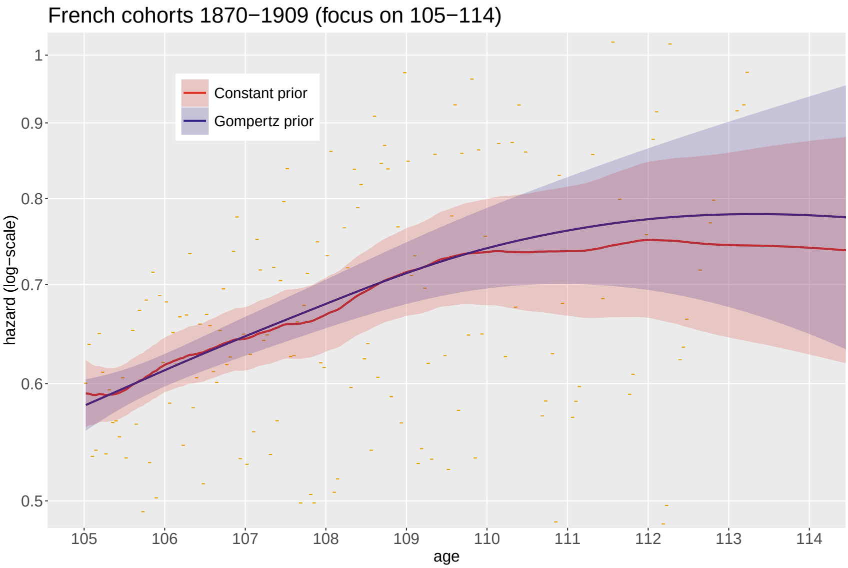

Including sex in a smooth setting

- A "proportional hazard" framework with a smooth relative risk function:

$$\begin{aligned} h_F(x) &= s(x) \\ h_M(x) &= h_F(x) \cdot e^{\delta(x)} \end{aligned}$$

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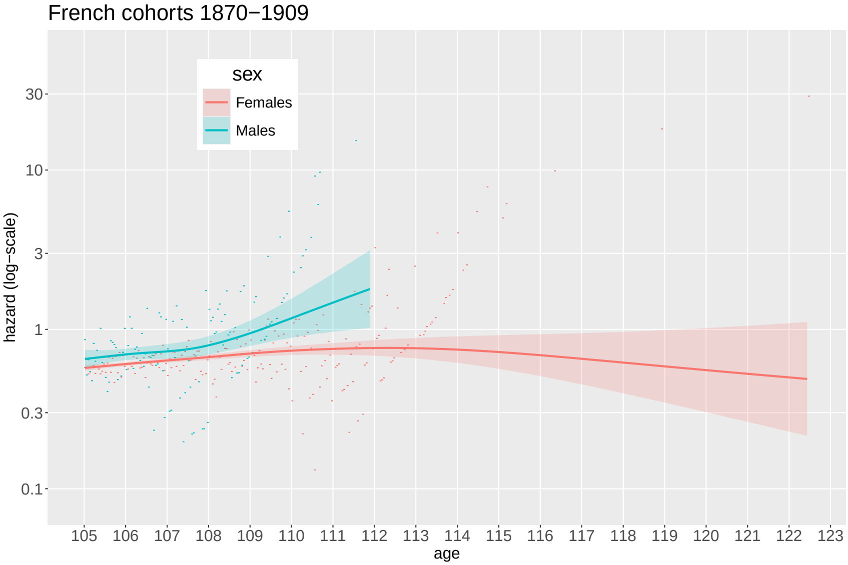
- "Some" adjustments in the previous equation

Log-mortality by sex



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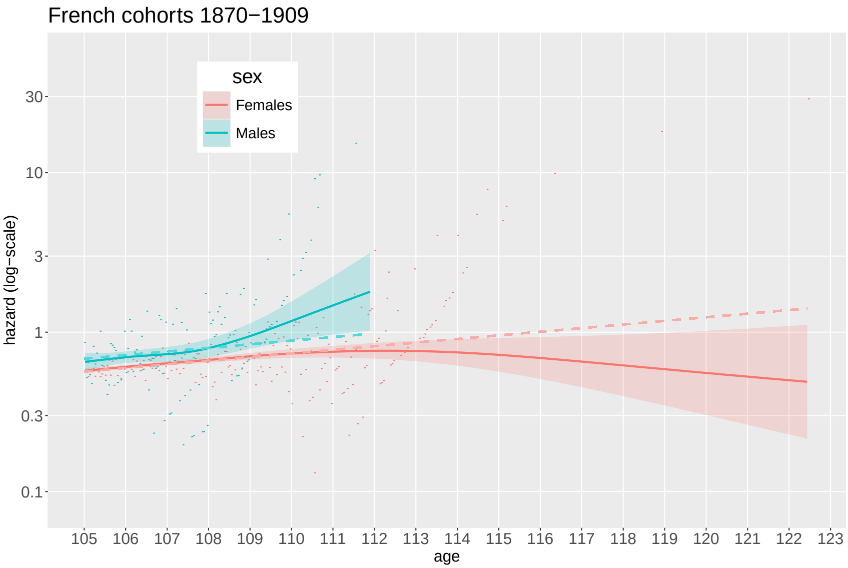
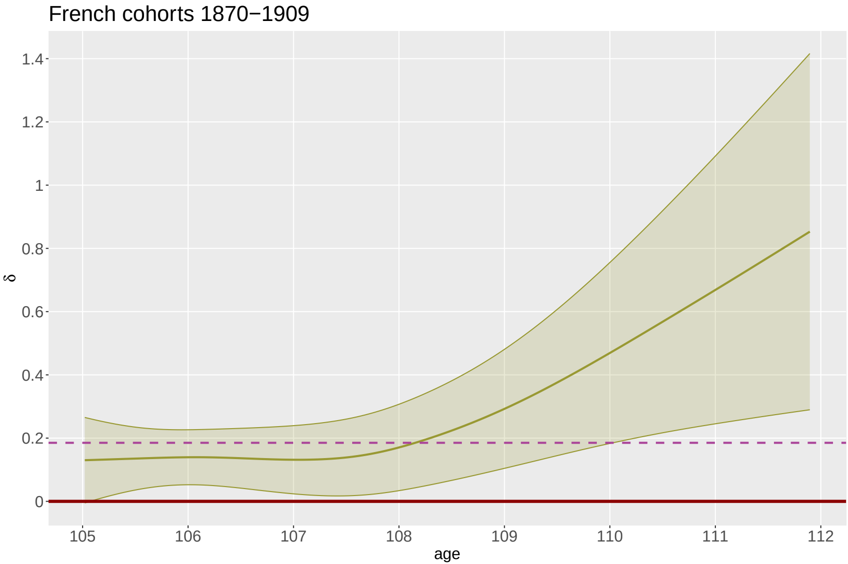
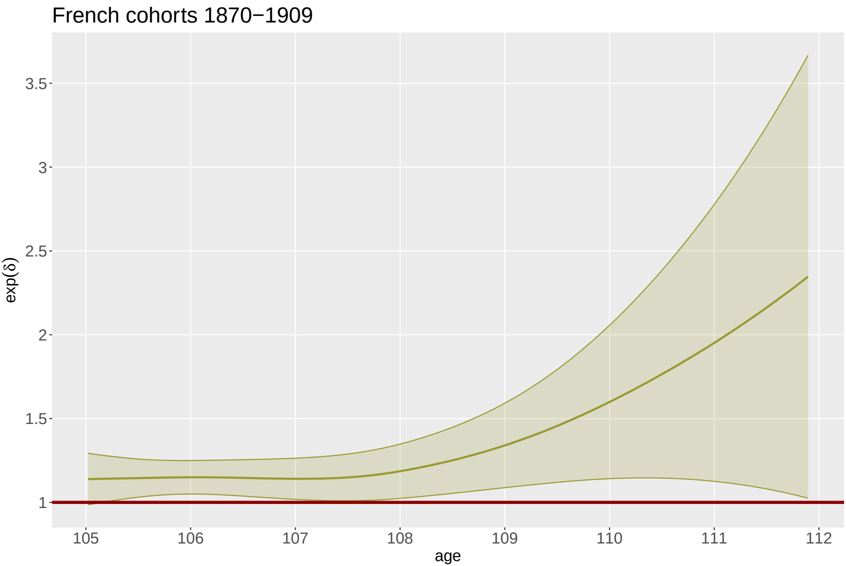


δ : sex difference in log-mortality



e^{δ} : hazard sex ratio



Concluding remarks

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  - reveals an increasing sex disadvantage from age 108
- Outlook:
  - Quality check of individual data for England & Wales
  - Ongoing efforts to gather data for Spain, Italy, Netherlands and Japan
  - Significant difficulties in accessing and publishing individual data: exploring the use of validated, aggregated data
  - Address right truncation in non-parametric models

Thanks for your attention.
Comments and questions?

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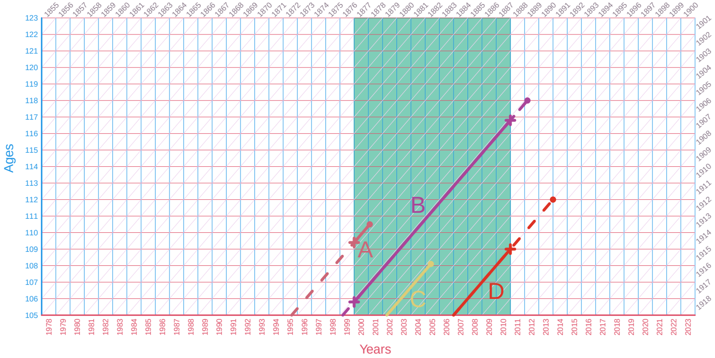
de la santé et de la recherche médicale

International Database on Longevity

- Aim to model, for instance, years 2000-2010



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B	1894.2	2012.2	118.0	2000.0	105.8	116.8	0
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$$\ell_A = P(X = 110.5 \mid X > 109.4) = \frac{f(110.5)}{S(109.4)}$$

$$\ell_B = P(X > 116.8 \mid X > 105.8) = \frac{S(116.8)}{S(105.8)}$$

$$\ell_C = P(X = 108.1 \mid X > 105.0) = \frac{f(108.1)}{S(105.0)}$$

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$$\ell_A = P(X = 5.5 \mid X > 4.4) = \frac{f(5.5)}{S(4.4)}$$

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$$\ell_D = P(X > 4.0) = S(4.0)$$

- We can condition everything on surviving to age 105

Modelling parametric models

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Model	Parameters estimates		95% CI	Log-likelihood	AIC
Constant	a	0.61445	[0.60332, 0.62559]	-19795.56	39593.13
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- Full dataset assuming cohorts born before 1909 are extinct:

Model	Parameters estimates		95% CI	Log-likelihood	AIC
Constant	a	0.61523	[0.60414, 0.62633]	-19797.68	39597.36
Gompertz	a	0.57564	[0.56146, 0.58981]	-19767.06	39538.13
	b	0.05063	[0.03835, 0.06291]		

- What if we extend our assumption beyond the 1909 cohort?

Including sex and/or cohort

- Check for sex differences in a proportional hazards setting:

$$h_i(x \mid \text{sex}_i, \text{cohort}_i) = h_0(x) \cdot \begin{cases} e^{\beta_{sex} \cdot \text{sex}_i} \\ e^{\beta_{cohort} \cdot \text{cohort}_i} \\ e^{\beta_{sex} \cdot \text{sex}_i + \beta_{cohort} \cdot \text{cohort}_i} \end{cases}$$

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	Covariate(s)	β	Estimated	95% CI	AIC
Constant	sex	β_{sex}	0.17573	[0.11317, 0.23828]	39571.10
	cohort	β_{cohort}	-0.00201	[-0.00414, 0.00013]	39595.89
	sex+cohort	β_{sex}	0.17563	[0.11271, 0.23855]	39569.51
		β_{cohort}	-0.00205	[-0.00417, 0.00007]	
Gompertz	sex	β_{sex}	0.18511	[0.12291, 0.24730]	39508.02
	cohort	β_{cohort}	-0.00032	[-0.00255, 0.00190]	39540.04
	sex+cohort	β_{sex}	0.18492	[0.12288, 0.24695]	39509.94
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- Gompertz baseline with only sex as a covariate selected by AIC

- Let's vectorize deaths and exposures for females and males

$$\boldsymbol{y} = [\boldsymbol{y}_F, \boldsymbol{y}_M]' \qquad \text{and} \qquad \boldsymbol{y} = [\boldsymbol{e}_F, \boldsymbol{e}_M]'$$

- Model linear predictor: $\boldsymbol{\eta} = [\boldsymbol{\eta}_F, \boldsymbol{\eta}_M]' = \boldsymbol{X}\boldsymbol{\beta}$, where

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0}_{m\times m} \\ \boldsymbol{I}_m & \boldsymbol{I}_m \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{\beta} = [\boldsymbol{\eta}_F, \boldsymbol{\delta}]'$$

- The iterative process to estimate $\boldsymbol{\beta}$ is given by

$$\tilde{\boldsymbol{\beta}} = (\boldsymbol{X}'\,\boldsymbol{W}\,\boldsymbol{X} + \boldsymbol{P})^{-1}\boldsymbol{X}'\,\boldsymbol{W}\,\boldsymbol{z}$$

where $\boldsymbol{z} = \frac{\boldsymbol{y}-\boldsymbol{\mu}\odot\boldsymbol{e}}{\boldsymbol{\mu}\odot\boldsymbol{e}} + \boldsymbol{\eta}$ and $\boldsymbol{W} = \texttt{diag}(\boldsymbol{\mu}\odot\boldsymbol{e})$

- The penalty term enforces smoothness of both $\boldsymbol{\eta}$ and $\boldsymbol{\delta}$:

$$\boldsymbol{P} = \begin{bmatrix} \lambda_{\eta_F}\boldsymbol{D}'\boldsymbol{D} & \\ & \lambda_{\delta}\boldsymbol{D}'\boldsymbol{D} \end{bmatrix}$$