Are $e^\dagger$ and Keyfitz H measures of inequality of ages at death?

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Construct validity

Does a statistic captures the concept it claims to measure?

- candidates: $e^{\dagger}$, Keyfitz H
- for construct: lifespan inequality in life table (LT) ages at death
- established measures: Gini Mean Difference (GMD), Theil index

We test conceptual and convergence validity.

Importance

1. Realist view: LT lifespan inequality exists independent of the measurement, the measure aims to capture it.
2. Operationalization and quantification of concepts shapes knowledge we produce
3. Value-laden concepts require transparency of the measurement

Organization of the talk

1. Background information: previous approximations
2. Conceptual validity 1: $e^{\dagger}$ as weighted deprivation
3. Conceptual 2: Deductive argument against $e^{\dagger}$
4. Conceptual 3: Keyfitz H as a measure of rectangularisation of survival curve
5. Convergent validity of $e^{\dagger}$ to Gini Mean Deviation and Keyfitz H to Gini

$$e^{\dagger} = \int_0^{\omega} f(x)\, e(x)\, dx$$

$$\text{Keyfitz } H = \frac{e^{\dagger}}{e(0)} = -\frac{1}{e(0)} \int_0^{\omega} \ell(x) \ln \ell(x)\, dx$$

for $l(0) = 1$

Previous approximations

1. Hakkert (1987) showed that:

$$H \approx \frac{2CV^2}{CV^2 + 1}, \tag{1}$$

CV is coefficient of variation.
It requires that, e.g., $e(0)^2 \approx 2\sigma^2$. For example, the higher $e(0)$, the smaller σ^2.

2. Shkolnikov et al. (2011) derived a formula for $e^\dagger$ similar to Gini Mean Deviation (GMD)

GMD measures deprivation of years lived for deaths at x as compared to longer lives

$$\mathrm{GMD} = \int_{x=0}^{\omega} \int_{y=x}^{\omega} (y-x) f(x) f(y) \,\mathrm{d}y \,\mathrm{d}x \tag{2}$$

$$e^\dagger = \int_{x=0}^{\omega} \frac{1}{\ell(x)} \int_{y=x}^{\omega} (y-x) f(x) f(y) \,\mathrm{d}y \,\mathrm{d}x$$

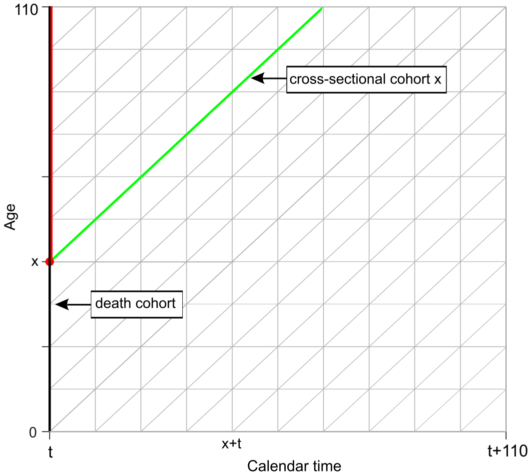
young age at death x $->$ high deprivation $->$ low weight $\frac{1}{\ell(x)}$.

Conceptual validity 1: Can a summary measure of deprivation, e.g., poverty, be constructed with such weights?

Deductive argument, preliminaries

Cohorts in period life table

- *Death cohort t*: deaths by age at time *t* in a stationary population
- *Cross-sectional cohort x, t*: those aged *x* at *t*, followed to death
- *Prevalent cohort*: all cross-sectional cohorts at *t*



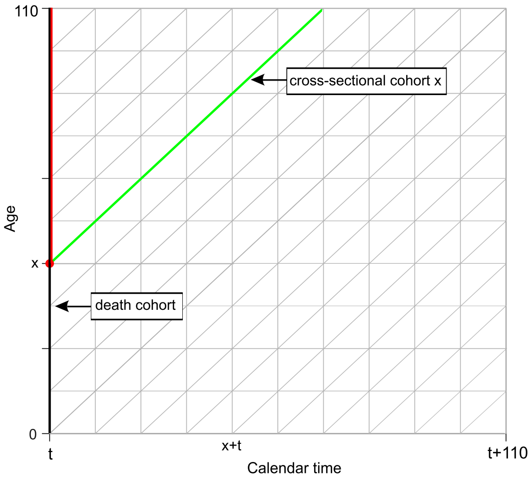
Deductive argument, preliminaries

In the cross-sectional cohort x, t, probability of survival to age y:

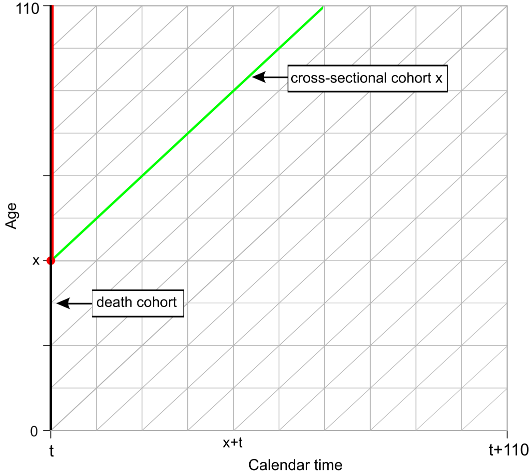
$$\ell(y,x) = \frac{\ell(y)}{\ell(x)} \qquad (3)$$

density of deaths at age y:

$$f(y,x) = \mu(y)\ell(y,x) = \mu(y)\frac{\ell(y)}{\ell(x)} = \frac{f(y)}{\ell(x)}$$



$e^\dagger$ for death cohort



$$e^\dagger = \int_{x=0}^{\omega} \int_{y=x}^{\omega} (y-x) f(x) \frac{f(y)}{\ell(x)} \, \mathrm{d}y \, \mathrm{d}x =$$

$$\int_{x=0}^{\omega} \int_{y=x}^{\omega} (y-x) f(x) f(y,x) \, \mathrm{d}y \, \mathrm{d}x$$

Construct validity 2: Deductive argument

A statistic of lifespan inequality compares ages at death of individuals **within the same cohort**.

$e^\dagger$ measures the average years of life deprivation of an individual in a death cohort to an **external reference**, i.e., ages at death of individuals in a corresponding cross-sectional cohort.

$e^\dagger$ is not a measure of inequality of life table ages at death.

Construct validity 3: Keyfitz H vs Theil index

Relative entropy, or Kullback–Leibler divergence, compares a study probability density distribution (G) to a benchmark distribution (Z)

$$D(G||Z) = \int_{x=0}^{\omega} g(x) \ln \frac{g(x)}{z(x)} \, \mathrm{d}x$$

Theil index is a relative entropy measure of divergence of distribution of **years lived per individual** to equal number of years lived per individual.

$$T = D(G||Z) = \int_{x=0}^{\omega} \frac{x}{e(0)} \ln \left(\frac{x}{e(0)} \right) f(x) \, \mathrm{d}x$$

Keyfitz H vs Theil

Kefitz H measures convergence of the distribution of **years lived per year of age** to equal number of years lived when everybody dies at max age.

$$H = \ln \frac{\omega}{e(0)} - \int_0^{\omega} w(x) \ln \frac{w(x)}{u(x)} \, \mathrm{d}x = \ln \frac{\omega}{e(0)} - D(W||U),$$

where W is the distribution of years across x: $w(x) = \frac{l(x)}{e(0)}$
U - fully rectangular distribution of years (apart from $x = \omega$), $\ell(x) = 1$ $\ln \frac{\omega}{e(0)}$ is *mortality shift* and $-D(W||U)$ is *level of compression*

Convergent validity

Empirical tools for testing convergence between $e^\dagger$ and GMD:

- **Correlation**: $e^\dagger$ and GMD:0.96, H and Gini:0.99, $\alpha < 0.01$
- **Reliability:** Intraclass correlation coefficient: ICC $\in$ [0.9;0.92] - excellent reliability (scale in Koo and Li (2016))
- **Agreement:** Bland–Altman limits of agreement: agreement is high except at very low $e(0)$
- Threshold age is consistently higher for $e^\dagger$ than for GMD and for Keyfitz H than Gini; the gap narrows with higher $e(0)$,

Data: United Nations Model Life tables (2024)

Summary

$e^\dagger$ and Keyfitz H have not passed the test for concept validity for measures of inequality in LT ages at death

- $e^\dagger$ puts higher value on short lives
- $e^\dagger$ measures deprivation to a benchmark outside of the study population
- Keyfitz H measures extent of rectangularisation
- the gap between threshold ages in $e^\dagger$ and GMD can be large

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Classification of lifespan inequality measures

Group	Example Measure	Interpretation
Range Measures	Interquantile Range	Spread of the middle 50%
Inter-individual differences	Gini Mean Difference	Mean difference between inidviduals in years lived
Individual/mean differences	Mean Average Deviation	Mean difference in years lived between individual and life expectancy
Entropy-based measures	Theil's index	Divergence from equal distribution of years of life between individuals

Source: Inequality measures classification from (Gakidou et al., 2000; Wilmoth and Horiuchi, 1999).

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Figure: Relative gap H - approximation of H

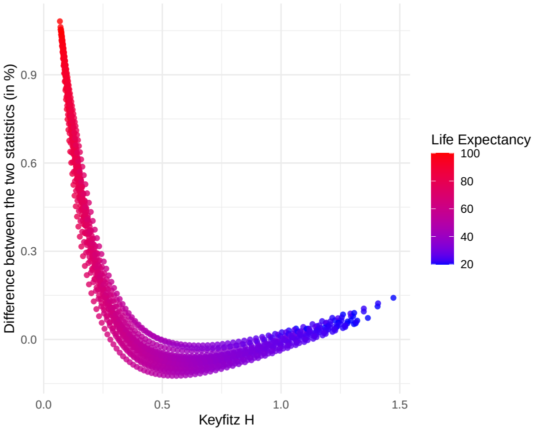

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Figure: Ranking, Males, e(0)=90

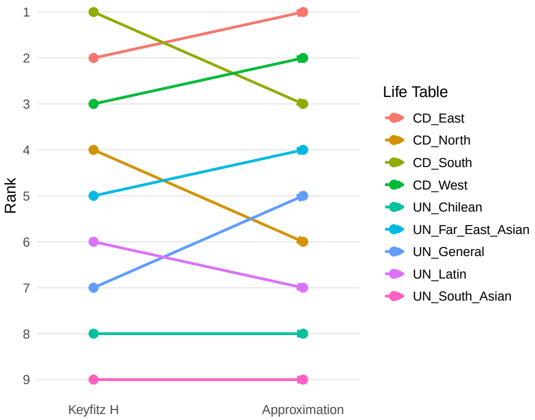

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Figure: Bland-Altman with 1.96 SD limits of agreement, $e^{\dagger}$ vs GMD

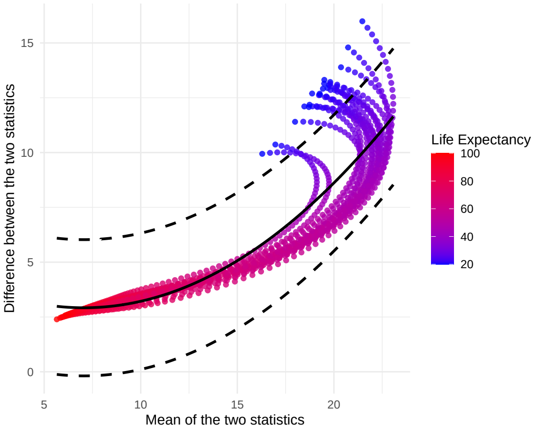

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Figure: Bland-Altman with 1.96 SD limits of agreement, H vs Gini

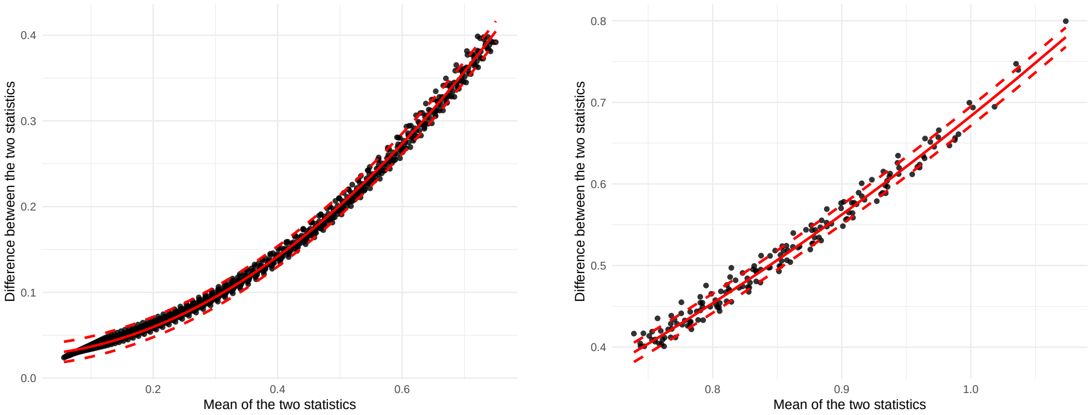

Figure: Bland–Altman with 1.96 SD limits of agreement: left, H vs Gini; right, H vs Approximation.

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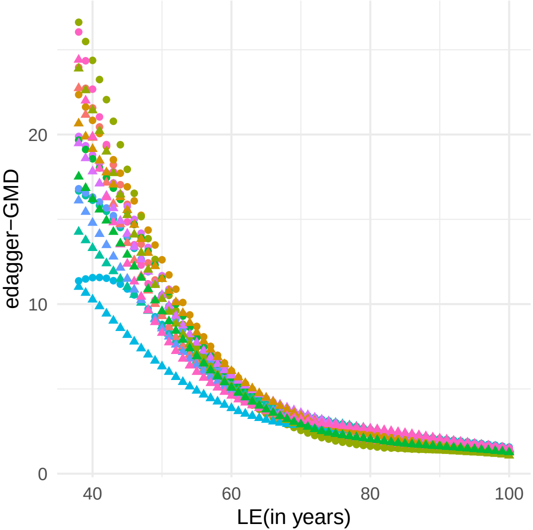
Data source: United Nations Model Life tables (2024).

Difference in threshold ages of $e^{\dagger}$ - GMD



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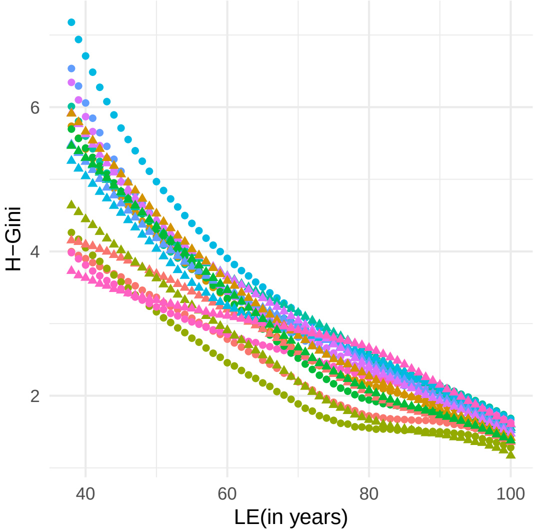
Difference in threshold ages of H - Gini



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